CONCOURS D'ADMISSION 2017 FILIÈRE UNIVERSITAIRE INTERNATIONALE
AUTUMN SESSION

MATHEMATICS

(Duration : 2 hours)

The three parts (Exercises I and II, Problem) are independant.
The use of computing devices is not allowed

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Exercise 1

Compute the value of the integral

$$\int_x^{x^2} \frac{dt}{t \ln t}.$$

for the values of the real number x for which it is defined. One may use a change of variable.

Exercise 2

We denote by $\mathcal{C}_0([0,+\infty))$ the set of bounded continuous functions from $[0,+\infty)$ to $\mathbb{R}$. We let a be a non-negative real number and $h \in \mathcal{C}_0([0,+\infty))$. In this exercise, we study the differential equation

$$f' + af = h \text{ denoted by } (E_a(h)),$$

where the unknown f is a differentiable function from $(0,+\infty)$ to $\mathbb{R}$.

Question 1. For any $a \geq 0$ determine all the solutions of the equation $f' + af = 0$.

Question 2. For $a \geq 0$ and $x \in (0,+\infty)$ we let

$$\varphi_a(x) = \int_0^x h(t)e^{+a(t-x)}dt.$$

Compute the derivative of φ_a.

Question 3. Using the function φ_a, determine all the solutions of $(E_a(h))$.

Question 4. Determine all the values of $a \geq 0$ such that for any $h \in \mathcal{C}_0([0,+\infty))$ all the solutions of the equation $(E_a(h))$ are bounded.

Question 5. Find a necessary and sufficient condition for h such that for any $a \geq 0$ all the solutions of the equation $(E_a(h))$ are bounded.

PROBLEM

GENERAL NOTATION. A vector $x \in \mathbb{R}^3$ is a column 3×1 matrix, which will also be represented as the transposed of an array 1×3 matrix, as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = {}^t(x_1, x_2, x_3).$$

We denote by $\{e^{(1)}, e^{(2)}, e^{(3)}\}$ the natural basis of $\mathbb{R}^3$, with $e_i^{(j)} = 1$ when $i = j$ and $e_i^{(j)} = 0$ when $i \neq j$.

In this problem, we study properties of the 3×3 real matrix

$$M = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.5 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

Question 1. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 3 & 3 \end{pmatrix}.$$

Question 2. Using Question 1., find the eigenvalues of the matrix M. They will be denoted by λ_1, λ_2 and λ_3, with $|\lambda_1| > \max(|\lambda_2|, |\lambda_3|)$, the choice between λ_2 and λ_3 being left to the candidate.

Question 3 Compute a non zero vector $\varepsilon^{(1)}$ in $\mathbb{R}^3$, eigenvector of M for the eigenvalue λ_1.

Question 4. Let $\varepsilon^{(2)}$ (*respectively* $\varepsilon^{(3)}$) be a non zero eigenvector for the eigenvalue λ_2 (*respectively* λ_3). Is the family $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}\}$ a basis of $\mathbb{R}^3$?

From now on we consider a vector

$$x^{(0)} = {}^t(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = y_1^{(0)}\varepsilon^{(1)} + y_2^{(0)}\varepsilon^{(2)} + y_3^{(0)}\varepsilon^{(3)},$$

we define by induction

$$x^{(n)} = Mx^{(n-1)}, \text{ for } n \geq 1$$

and we write

$$x^{(n)} = {}^t(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) = y_1^{(n)}\varepsilon^{(1)} + y_2^{(n)}\varepsilon^{(2)} + y_3^{(n)}\varepsilon^{(3)}.$$

Question 5. Show that for any $j \in \{1,2,3\}$, the sequence $\left(y_j^{(n)}\right)_{n\geq 0}$ is convergent and determine its limit.

Question 6. Deduce from the previous question that for any $j \in \{1, 2, 3\}$, the sequence $\left(x_j^{(n)}\right)_{n \geq 0}$ is convergent and determine its limit.

Question 7. Show that for any integer n the sum of the entries of each column of the matrix M^n is equal to 1.

Question 8. Show that the sequence of the powers of the matrix M (that is to say the sequence $(M^n)_n$) converges to a matrix all the columns of which are equal. We recall that a sequence of matrices $B^{(n)} = (b_{i,j}^{(n)})_{i,j}$ converges if for any pair (i, j), the sequence $(b_{i,j}^{(n)})_n$ converges.

Question 9. Determine explicitly the limit of the sequence $(M^n)_n$.