ÉCOLE POLYTECHNIQUE UNIVERSITÉ PARIS-SACLAY ESPCI PARIS ISAE SUPAERO

CONCOURS D'ADMISSION 2018 **Session d'automne**

FILIÈRE UNIVERSITAIRE INTERNATIONALE

MATHEMATICS

(Duration : 2 hours)

The three parts (Exercises 1 and 2, Problem) are independant.
The use of computing devices is not allowed

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Exercise 1

Let a and b be two real numbers with $a < b$ and let f be a differentiable function from $[a, b]$ to $\mathbb{R}$ such that $f(a) = f(b) = 0$. We assume that f' is bounded on $[a, b]$ and let $K = \sup_{t\in[a,b]} |f'(t)|$.

1.1 Show that one has

$$\forall t \in [a, b] : |f(t)| \leq K(t - a) \quad \text{and} \quad \forall t \in [a, b] : |f(t)| \leq K(b - t). \tag{1}$$

1.2 Compute $\int_a^{(a+b)/2} (t - a)dt$.

1.3 Using the previous questions, show that one has

$$\left| \int_a^b f(t)dt \right| \leq K \frac{(b-a)^2}{4}.$$

Exercise 2

We denote by $\mathcal{M}_3(\mathbb{R})$ the vector space of the real 3×3 matrices. We consider the following four matrices

$$A = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 1 & 0 \\ -2 & 0 & 4 \end{pmatrix}, \ P = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

2.1 Let $B \in \mathcal{M}_3(\mathbb{R})$. Show that the set

$$\mathcal{C}(B) = \{M \in \mathcal{M}_3(\mathbb{R}) \colon BM = MB\}$$

is a linear subspace of $\mathcal{M}_3(\mathbb{R})$.

2.2 Determine $\mathcal{C}(D)$.

2.3 Compute the products PQ and PDQ.

2.4 Let Φ_P the map from $\mathcal{M}_3(\mathbb{R})$ to $\mathcal{M}_3(\mathbb{R})$ which associates to any matrix N the matrix $P^{-1}NP$. The map Φ_P is a linear map from $\mathcal{M}_3(\mathbb{R})$ to $\mathcal{M}_3(\mathbb{R})$, the proof of this point is not asked for. Determine the composition product $\Phi_{P^{-1}} \circ \Phi_P$ which associates to any matrix $N \in \mathcal{M}_3(\mathbb{R})$ the matrix $\Phi_{P^{-1}}(\Phi_P(N))$.

2.5 By considering the image of $\mathcal{C}(A)$ by Φ_P and using the previous questions, determine the dimension of $\mathcal{C}(A)$.

PROBLEM

The aim of the problem is to study the non-linear recurrence sequence $(u_n)_{n\geq 0}$ defined by

$$u_0 = 0,\ u_1 = L \text{ and, for } n \geq 0 \colon u_{n+2} = \sqrt{u_{n+1}} + \sqrt{u_n}, \tag{2}$$

where L is a positive real number. We denote by m the minimum of the two numbers 1 and L and by M the maximum of those two numbers.

In Part A, we study the first properties of the sequence $(u_n)_{n\geq 0}$; in Part B, which is independent of Part A, we study an auxiliary linear recurrence sequence and in Part C we end the study of the sequence $(u_n)_{n\geq 0}$.

PART A, First properties of the sequence $(u_n)_{n\geq 0}$

A.1 Show that for $n \geq 1$, one has $m \leq u_n \leq 4M$.

A.2 Assuming that the sequence $(u_n)_{n\geq 0}$ converges, find the value of its limit.

PART B, An auxiliary linear recurrence sequence

In this part, K denotes a real number in the interval $(0, 1/2)$.

B.1 Let f be the quadratic polynomial defined by $f(x) = x^2 - Kx - K$. Compute $f(-1)$, $f(0)$ and $f(1)$ and show that f has two real roots (or in other words, the equation $f(x) = 0$ has two real solutions), which are denoted as $c < C$.

B.2 Let H be a positive number. We consider a sequence $(v_n)_{n\geq 0}$ satisfying

$$0 \leq v_0 \leq H,\ 0 \leq v_1 \leq HC \text{ and, for } n \geq 0 \colon v_{n+2} = K\left(v_{n+1} + v_n\right). \tag{3}$$

Show that for any $n \geq 0$ one has $0 \leq v_n \leq HC^n$.

B.3 Show that the sequence $(v_n)_{n\geq 0}$ converges.

PART C, End of the study of the sequence $(u_n)_{n\geq 0}$

C.1 Show that for $n \geq 0$ one has

$$|u_{n+2} - 4| \leq \frac{|u_{n+1} - 4|}{\sqrt{u_{n+1}} + 2} + \frac{|u_n - 4|}{\sqrt{u_n} + 2}. \tag{4}$$

C.2 Let $K = 1/(\sqrt{m}+2)$ and $(w_n)_{n\geq 0}$ be a linear recurrence sequence such that $w_0 \geq 4$, $w_1 \geq |L - 4|$ and, for $n \geq 2$: $w_{n+2} = K\,(w_{n+1} + w_n)$. Show that for any $n \geq 0$, one has $|u_n - 4| \leq w_n$.

C.3 Using the previous questions, show that the sequence $(u_n)_{n\geq 0}$ converges.