Problem

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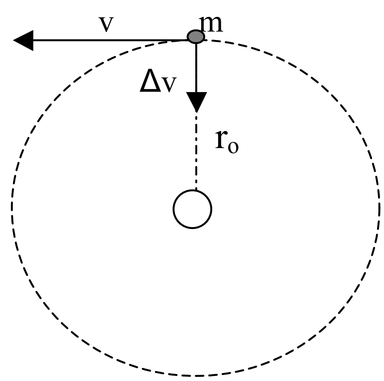
We consider a geosynchronous communication satellite, i.e. a satellite whose revolution period is equal to the period of rotation of the Earth around itself. Tha mass of the satellite is m and it is placed in an equatorial circular orbit of radius r_o. These satellites have an “apogee engine” which provides the tangential thrusts needed to reach the final orbit.

We give: Earth radius R_T=6.37.10^{6}m, Earth surface gravity g=9.81m.s^{-2}, and the period of rotation of the Earth around itself is T_o=24h.

1.1 Give the analytical expression of the orbit radius r_o as a function g, R_T, T_o and then velocity v_o of the satellite as a function g, R_T, and r_o.

1.2 Obtain the expression of its angular momentum L_o and mechanical energy E_o as a function of v_o, m, g and R_T.



Once this geosynchronous circular orbit has been reached, an error by the ground controllers causes the apogee engine to be fired again. The thrust happens to be directed towards the Earth and causes and unwanted velocity variation Δv to be imparted on the satellite. We characterize this boost by the parameter $\beta=\Delta v/v_o$. The duration of the engine burn can be considered as instantaneous.

Suppose $\beta<1$.

2.1 Determine the parameters of the new orbit, semi-latus-rectum l, and eccentricity ε in term of r_o and β.

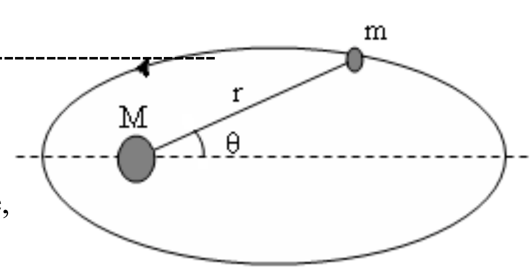
2.2 Calculate the angle between the major axis of the new orbit and the position vector at accidental misfire.

2.3 Give the analytical expression of the perigee r_{min} and apogee r_{max} distances to the Earth center, as functions of r_o and β.

2.4 Determine the period of the new orbit T as a function of T_o and β.

2.5 Calculate the minimum boost parameter β_{min}, needed to escaped for the satellite to escape Earth gravity.

2.6 Determine in this case the closet approach of the satellite to the Earth centre in the new trajectory, r'_{min} as a function of r_o.



Hint

In polar coordinate, taking the origin at focus point, the general equation of a conic (ellipse, parabola or hyperbola) can be written as:

$$r(\theta)=\frac{l}{1-\varepsilon\cos\theta}$$

where l is the semi-latus-rectum and ε the eccentricity.
In terms of constants of motion:

$$l = \frac{L^2}{GMm^2} \qquad and \qquad \varepsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

where G is the Newton constant, L is the modulus of the angular momentum of the orbiting mass, with respect of the origin, and E is its mechanical energy, with zero potential at infinity.

We may have the following cases:

i. If $0 \le \varepsilon < 1$, the curve is an ellipse

ii. If ε=1, the curve is a parabola

iii. If ε>1, the curve is a hyperbola.

Solution

1.1

$$\left.\begin{aligned} G\frac{M_T m}{r_0^2} &= m\frac{v_0^2}{r_0} \\ v_0 &= \frac{2\pi r_0}{T_0} \\ g &= \frac{GM_T}{R_T^2} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} r_0 &= \left(\frac{g R_T^2 T_0^2}{4\pi^2}\right)^{1/3} \quad \Rightarrow \quad r_0 = 4.22\cdot 10^7 \text{ m} \\ v_0 &= R_T\sqrt{\frac{g}{r_0}} \quad \Rightarrow \quad v_0 = 3.07\cdot 10^3 \text{ m/s} \end{aligned}\right.$$

1.2

$$L_0 = r_0\, m\, v_0 = \frac{g R_T^2}{v_0^2} m\, v_0 \quad \Rightarrow \quad \boxed{L_0 = \frac{m\, g\, R_T^2}{v_0}}$$

$$E_0 = \frac{1}{2}mv_0^2 - G\frac{M_T m}{r_0} = \frac{1}{2}mv_0^2 - \frac{g R_T^2 m}{r_0} = \frac{1}{2}mv_0^2 - mv_0^2 \quad \Rightarrow \quad \boxed{E_0 = -\frac{1}{2}mv_0^2}$$

2.1

The value of the *semi-latus-rectum* l is obtained taking into account that the orbital angular momentum is the same in both orbits. That is

$$l = \frac{L_0^2}{G M_T m^2} = \frac{m^2 g^2 R_T^4}{v_0^2}\frac{1}{g R_T^2 m^2} = \frac{g R_T^2}{v_0^2} = r_0 \quad \Rightarrow \quad \boxed{l = r_0}$$

The eccentricity value is

$$\varepsilon^2 = 1 + \frac{2 E L_0^2}{G^2 M_T^2 m^3}$$

where E is the new satellite mechanical energy

$$E = \frac{1}{2} m \left(v_0^2 + \Delta v^2 \right) - G \frac{M_T m}{r_0} = \frac{1}{2} m \Delta v^2 + E_0 = \frac{1}{2} m \Delta v^2 - \frac{1}{2} m v_0^2$$

that is

$$E = \frac{1}{2} m v_0^2 \left(\frac{\Delta v^2}{v_0^2} - 1 \right) = \frac{1}{2} m v_0^2 \left(\beta^2 - 1 \right)$$

Combining both, one gets $\boxed{\varepsilon = \beta}$

This is an elliptical trajectory because $\varepsilon = \beta < 1$.

2.2

The initial and final orbits cross at P, where the satellite engine fired instantaneously (see Figure 1). At this point

$$r(\theta = \alpha) = r_0 = \frac{r_0}{1 - \beta \cos \alpha} \quad \Rightarrow \quad \boxed{\alpha = \frac{\pi}{2}}$$

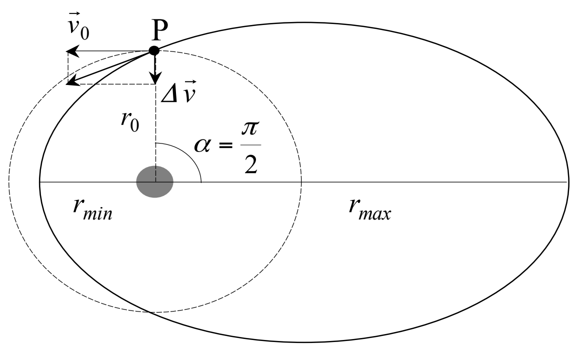


Figure 1

2.3

From the trajectory expression one immediately obtains that the maximum and minimum values of r correspond to $\theta = 0$ and $\theta = \pi$ respectively (see Figure 4). Hence, they are given by

$$r_{max} = \frac{l}{1 - \varepsilon} \qquad r_{min} = \frac{l}{1 + \varepsilon}$$

that is:

$$\boxed{r_{\max} = \frac{r_0}{1 - \beta}} \quad \text{and} \quad \boxed{r_{\min} = \frac{r_0}{1 + \beta}}$$

The distances r_{max} and r_{min} can also be obtained from mechanical energy and angular momentum conservation, taking into account that $\vec{r}$ and $\vec{v}$ are orthogonal at apogee and at perigee

$$E = \frac{1}{2} m v_0^2 \left(\beta^2 - 1 \right) = \frac{1}{2} m v^2 - \frac{g R_T^2 m}{r}$$

$$L_0 = \frac{m g R_T^2}{v_0} = m \, v \, r$$

What remains of them, after eliminating *v*, is a second-degree equation whose solutions are r_{max} and r_{min}.

2.4

By the Third Kepler Law, the period *T* in the new orbit satisfies that

$$\frac{T^2}{a^3} = \frac{T_0^2}{r_0^3}$$

where *a,* the semi-major axis of the ellipse, is given by

$$a = \frac{r_{max} + r_{min}}{2} = \frac{r_0}{1-\beta^2}$$

Therefore

$$\boxed{T = T_0\left(1-\beta^2\right)^{-3/2}}$$

2.5

Only if the satellite follows an open trajectory it can escape from the Earth gravity attraction. Then, the orbit eccentricity has to be equal or larger than one. The minimum boost corresponds to a parabolic trajectory, with $\varepsilon = 1$

$$\varepsilon = \beta \qquad \Rightarrow \qquad \boxed{\beta_{esc} = 1}$$

This can also be obtained by using that the total satellite energy has to be zero to reach infinity ($E_p = 0$) without residual velocity ($E_k = 0$)

$$E = \frac{1}{2} m v_0^2 \left(\beta_{esc}^2 - 1\right) = 0 \quad \Rightarrow \qquad \beta_{esc} = 1$$

This also arises from $T = \infty$ or from $r_{max} = \infty$.

2.6

Due to $\varepsilon = \beta_{esc} = 1$, the polar parabola equation is

$$r = \frac{l}{1-\cos\theta}$$

where the semi-latus-rectum continues to be $l = r_0$. The minimum Earth - satellite distance corresponds to $\theta = \pi$, where

$$\boxed{r'_{min} = \frac{r_0}{2}}$$

This also arises from energy conservation (for $E = 0$) and from the equality between the angular momenta (L_0) at the initial point P and at maximum approximation, where $\vec{r}$ and $\vec{v}$ are orthogonal.