ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION AUTOMNE 2015

PHYSICS

(Duration : 2 hours)

The two problems are independent. If you are not able to solve a question, we advise you to assume the result of that question and to move to the next one. The different questions are, to a large extent, independent of one another.
The figures and captions are part of the text : they should be read carefully.

The use of electronic calculators is forbidden.

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1 Surface Tension and Capillarity

Surface tension appears at the interface between a liquid phase (such as water) and a gas phase (such as air) due to the molecular attraction amongst the molecules of the liquid. The net effect is that molecules close to the surface of a liquid are subject to an inward force similar to the tension in a stretched membrane. Because of surface tension, the interface between the liquid and the gas has a tendency to minimize its area. Hence, in order to increase the area A of the interface by an amount dA, one has to perform a work $\delta W = \sigma dA$. The physical quantity σ is the surface tension, with units J.m^{-2}.

For water, we shall take $\sigma \simeq 72.5 \times 10^{-3}$ J.m^{-2} at temperature $T = 20°$ C.

1. We consider a spherical droplet of liquid of radius R at equilibrium with the surrounding air at atmospheric pressure P_0. We call P_{in} the pressure inside the droplet. Show that

$$P_{\text{in}} - P_0 = \frac{2\sigma}{R}$$

This formula is know as Laplace's law.

Hint : Suppose that the radius of the droplet increases by an amount dR. Calculate the work due to pressure forces and identify it to the increase of surface energy.

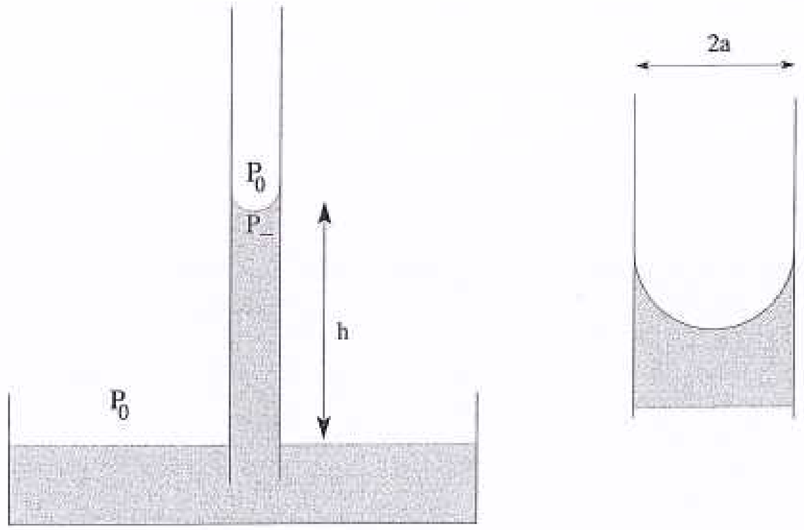

FIGURE 1 – A capillary tube of small diameter $2a$ and open at its two ends is immersed in a basin of water. Some liquid rises in the tube thanks to surface tension forces. The external pressure is the atmospheric pressure P_0. The pressure just below the interface is denoted by P_-. A blow-up of the picture is drawn on the right : the shape of the interface is taken to be a hemisphere.

For a droplet of water, calculate the required value of R to have an internal pressure that exceeds the atmospheric pressure by 1% ? ($P_0 \simeq 10^5 \mathrm{N/m^2}$).

2. We study the rise of water due to capillarity in a thin cylindrical tube made of clean glass of radius a immersed in a basin full of water. We suppose that the interface has the shape of a full hemisphere (see Figure 1). Using Laplace's law, show that the height h of water in the tube is given by

$$h = \frac{2\sigma}{\rho g a}$$

where $\rho = 1$ kg/l is the density of water and $g = 9.8$ m/s^2 is the standard acceleration due to gravity.

Calculate the value of h for $a = 0.5$ mm.

3. Consider the more general case where the interface makes an angle ψ with the wall of the tube (see Figure 2). The value of the contact angle ψ is determined by the properties of the liquid, the gas and the material of the tube. Show that the height h is now given by

$$h = \frac{2\sigma}{\rho g a} \cos\psi$$

Hint : show that the radius of curvature of the interface is given by $R = a/\cos\psi$.

What happens when $\psi > \frac{\pi}{2}$? Do you know a liquid for which this is the case ?

4. We now study some thermodynamical properties of surface phenomena. It can be shown that the surface tension is a function of temperature $\sigma(T)$. The amount of work required to

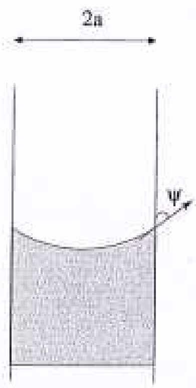

FIGURE 2 – More general interface : the boundary angle between the capillary tube and the surface of the water is given by ψ. Note that the picture drawn in Figure 1 corresponds to $\psi = 0$.

increase the interface by dA is

$$\delta W = \sigma(T) dA$$

(i) From the first and the second principles of Thermodynamics, show that the elementary amount of heat transferred to the interface when its temperature is increased by dT and its surface by dA is given by

$$\delta Q = C_A dT - T\frac{d\sigma(T)}{dT} dA$$

where C_A is the specific heat per unit area.

Hint : Write $\delta Q = C_A dT + l dA$ and give the expressions for the variations of the internal energy U and of the entropy S (discard the contribution of pressure forces). Use the fact that dU and dS are exact differentials to prove $l = -T\frac{d\sigma(T)}{dT}$.

The surface tension of water changes linearly with temperature from 75×10^{-3}N/m at 5°C to 70×10^{-3}N/m at 35°C. Calculate the heat entering a water surface when its area is increased by unity under isothermal conditions at 27°C.

(ii) Recalling that U and S are extensive with respect to A and writing $C_A = Ac(T)$, prove that

$$c(T) = -T\frac{d^2\sigma(T)}{dT^2}$$

(iii) Deduce the following formulas for the thermodynamic potentials :

$$\begin{aligned} U &= U_0 + A(\sigma - T\frac{d\sigma(T)}{dT}) \\ S &= S_0 - A\frac{d\sigma(T)}{dT} \\ F &= U - TS = F_0 + A\sigma(T) \end{aligned}$$

Using extensivity once again, show that $U_0 = S_0 = F_0 = 0$.

(iv) An atomizer produces water droplets of diameter $2R = 10^{-5}$cm. A cloud of droplets at 35°C coalesces to form a single drop of mass 1g. Estimate the temperature of the drop if there is no heat exchange with the surroundings. What is the increase of entropy in the process?

Hint : Calculate the variation of the total surface. Because there is no heat exchange and no external work performed, the total energy is conserved : the loss of surface energy of the cloud contributes to the internal energy of the single drop. We recall that the bulk heat capacity of water is $C_v \simeq 4.18$ J K^{-1} g^{-1}.

2 Adiabatic invariants in classical mechanics

In this problem we study the motion of a mechanical particle in one-dimension, governed by the following equation

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\frac{\partial\mathcal{U}(x)}{\partial x}\,,$$

where $\mathcal{U}(x)$ represents an external potential. The motion of the particle is determined by the initial conditions at time $t = 0$:

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0\,.$$

2.1 Phase Portraits

The phase space of the system is a two dimensional space with coordinates $(x, p = m\dot{x})$. A phase portrait is a geometric representation of a trajectory of the system starting from arbitrary initial conditions (x_0, mv_0) in the phase plane.

1. For a free particle, show that the phase portrait is a set of horizontal lines.

2. Study the case of a particle confined in a box of size L in the region $-L/2 \leq x \leq L/2$. The particle bounces elastically on the walls of the box. Draw the phase portrait for a given value of the velocity v of the particle. What is the value of the surface enclosed by the trajectory in the phase space ?

3. When $\mathcal{U}(x) = \frac{1}{2}m\omega^2x^2$, we have a harmonic oscillator : show that the trajectories in the phase space are ellipses. Calculate the values of the axis of the ellipse as a function of A, the amplitude of the oscillator. Prove that the total energy E satisfies

$$E = \frac{1}{2}m\omega^2A^2$$

and that the area under the trajectory of total energy E is given by E/ν, where ν is the frequency of the oscillator.

Plot qualitatively the trajectories when a damping force $-\gamma\dot{x}$ is added.

2.2 Particle in a box

We consider a particle of mass m and speed v that bounces elastically against the walls of a box of size L ; the walls are located at $x = \pm L/2$. The dynamics is confined in one dimension.

We suppose that the box is enlarged very slowly from L to $2L$ by moving the walls apart : the right hand wall and the left hand wall moving at speeds $\dot{L}/2$ and $-\dot{L}/2$, respectively.

1. Find the variation Δv of the speed of the particle after each collision and calculate the time τ between two successive collisions. What is the variation ΔL of the length L during the same time interval τ ? Deduce that Δv and ΔL satisfy

$$\frac{\Delta v}{v} + \frac{\Delta L}{L} = 0$$

Conclude that the value of vL remains constant. This quantity vL is called an *adiabatic invariant.* What is its dimension ? What does it represent in the phase space ? What is the value of v at the end of the process ?

2. From this adiabatic invariant, obtain the equation $PV^{5/3}$ = Constant for the adiabatic curves in a mono-atomic perfect gas.

Hint : Relate v to temperature using equipartition of energy and L to the total volume V. What happens for a diatomic gas ?

3. Can you relate this adiabatic invariant to the quantization of the energy levels of a particle in a box ?

2.3 Harmonic oscillator

We consider a pendulum, i.e. a mass m attached to a thin wire of length l and oscillating with a small amplitude $\theta(t)$. The motion is supposed to be harmonic.

1. Show that the motion is given by

$$\theta = \theta_{\text{max}} \cos \omega t \quad \text{and} \quad \dot{\theta} = -\omega\theta_{\text{max}} \sin \omega t$$

where $\omega = \sqrt{g/l}$ and θ_{max} is the maximal amplitude.

Calculate the total energy E as a function of $m, g, l, \theta_{\text{max}}$.

2. Show that the force T due to the tension of the thread is given by

$$T = mg(1 - \frac{\theta^2}{2}) + ml\dot{\theta}^2$$

Calculate the average value $\langle T \rangle$ of the tension during one period of oscillation.

We now suppose that the length l of the wire changes with time by a very slow process such that $\dot{l} \ll l\omega$. To give a precise picture, we suppose that an operator (for example, a person) pulls very slowly the wire and shortens it (see Figure 3). For the process to be very slow, the force applied by the operator exceeds the tension T of the wire only by an infinitesimal (and negligible) amount.

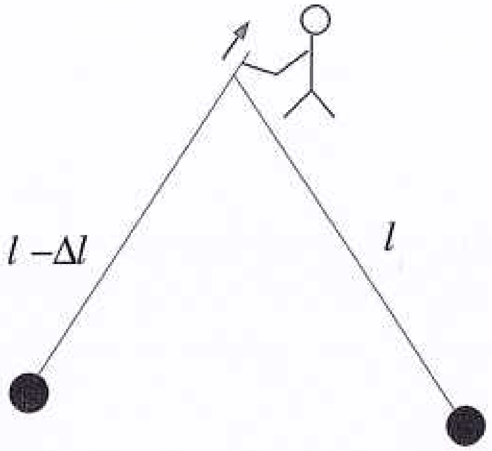

FIGURE 3 – An operator is pulling gently the wire of the pendulum. As a result, the length of the wire is becoming smaller by a very slow process.

3. Calculate the average work done by the operator, that is the work done by the average tension $\langle T \rangle$ when the length varies by Δl. Why are we allowed to replace T by its average ?

4. Show that the variation of the energy of the pendulum when l varies by Δl is given by a sum of two contributions : a term giving the variation of potential energy and a term related to the variation of the amplitude. Show that the second term satisfies

$$\frac{\Delta E}{E} = -\frac{1}{2}\frac{\Delta l}{l}\,.$$

5. Deduce from the previous question that the quantity E/ν, where ν is the frequency of the oscillator, is invariant throughout the process. This is an adiabatic invariant.

6. Suppose that the length l is reduced to $l/2$ by a very slow process. What is the maximal amplitude of the oscillations at the end of the process ?

7. Give an interpretation of the adiabatic invariant E/ν in the phase space (see question 3 in section **3.1**. What are the dimensions of E/ν ? Do you know a fundamental physical constant having the same dimensions ? Can you relate E/ν and the quantization of the energy levels of a harmonic oscillator ?

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