QCM (Dara)

1. Mr. and Mrs. Brown have two children, boy or girl. The probabilities that they have a son or a daughter are equal. Find probability that Mr. and Mrs. Brown have two sons, knowing that one of them is a boy.

a. 1/4
b. 1/2
c. 1/3
d. a, b , and c are not the answer

Prove: There are 4 configurations possible (S, S), (S, D), (D, S), (D, D). Note: S for the event of "Son" and D for the event of "Daughter". There are 3 configurations that one of them is a boy: (S, S), (S, D), (D, S). Thus, the probability that that Mr. and Mrs. Brown have two sons, knowing that one of them is boy is 1/3.

2. Tom and his 3 friends, who were born in the same mouth of October have a small birthday party in his house. Find probability that at last two of them we were born in the same day.

a. 0.016
b. 0.5
c. 0.002
d. a, b , and c are not the answer

Prove: Suppose that P_n is probability that at lest two of n people were born in the same day ($n \leq 31$). That probability is

$$P_n = 1 - \frac{31\times 30\times \cdots \times (31-n+1)}{(31)^n}$$

Thus $P_4 = 0.182$

3. Let $R_\theta = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$ be a rotation matrix that belongs to $\mathcal{M}_2(\mathbb{R})$. Suppose $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$,

a. $Sp(R_\theta) = \{-1, 1\}$
b. $Sp(R_\theta) = \emptyset$
c. $Sp(R_\theta) = \{1\}$
d. $Sp(R_\theta) = \mathbb{R}$

Prove: The characteristic polynomial of R_θ is

$$P_\theta(X) = X^2 - 2cos(\theta)X + 1 = \big(X - cos(\theta)\big)^2 + sin^2(\theta)$$

This polynomial $P_\theta(X)$ doesn't have real roots for $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$. Thus, $Sp(R_\theta) = \emptyset$ for $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$ ($R_\theta = \pm Id$ for $\theta \in \pi\mathbb{Z}$).

4. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k.

a. All coefficients of f are divisible by 5
b. At last one of coefficients of f are not divisible by 5
c. f has real roots divisible by 5
d. a, b and c are not true

Prove: Let $f(x) = ax^2 + bx + c$. For $x = 0, 5|f(0) = c$. For $x = \pm 1$, we obtain that $5|f(1) = a + b + c$ and $5|f(-1) = a - b + c$. Then $5|f(1) + f(-1) - 2f(0) = 2a$ and $5|f(1) - f(-1) = 2b$. Thus all coefficients of f are divisible by 5. We don't have enough information to answer questions "c".

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. This imply that :

a. $f(x) = ax + b$ for some real numbers a and b.
b. f is not an affine function.
c. f is a constant function.
d. a, b and c are not true.

Prove: The function $f(x) = e^x$ also has this property $\left(ce^x = e^{x+\ln(c)}\right)$

6. Calculate

$$I = \lim_{n\to+\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}$$

a. $4/e$
b. e
c. 0
d. $+\infty$

Prove: We have

$$\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} = \exp\left(\frac{1}{n}\sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right)\right)$$

For all $n \in \mathbb{N}^*$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right) = \int_0^1 \ln(1+t)\, dt = 2\ln 2 - 1$$

Thus $I = 4/e$

7. Let $f : [0,1] \longrightarrow \mathbb{R}$,

$$x \longmapsto f(x) = \begin{cases} xE\left(\frac{1}{x}\right) & if \;\; x \neq 0 \\ 1 & if \;\; x = 0 \end{cases}$$

a. $\forall\, x \in [0,1], f\big(f(x)\big) = f(x) + 1$
b. $\forall\, x \in [0,1], f\big(f(x)\big) = f(x)$
c. $\forall\, x \in [0,1], f\big(f(x+1)\big) = f(x)$

d. a, b and c are not true

Prove:

- If $x = 0$, $f(f(x)) = f(x)$ or $f(f(x+1)) = f(x)$
- If $x \in]1/2, 1]$, so $1 \le 1/x < 2 \implies E(1/x) = 1$

$$f(x) = xE\left(\frac{1}{x}\right) = x \ , \qquad f(f(x)) = x$$

- If $x \in]0, 1/2]$, since $1/x - 1 < E(1/x) \le 1/x$, so $1/2 < f(x) \le 1$. Thus $f(f(x)) = f(x)$

8. Determine, $n \in \mathbb{N}^*$

$$\lim_{n\to+\infty} \int_0^1 \sqrt{1 - x^n}\, dx$$

a. $+\infty$
b. 1
c. 0
d. a, b and c are not answers

Prove:

$$\left|\int_0^1 \sqrt{1 - x^n}\, dx - 1\right| = \left|\int_0^1 \frac{x^n}{\sqrt{1 + x^n - 1}} dx\right| \le \int_0^1 x^n\, dx = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0$$

9. Let M be a $n{\times}n$ matrix, $M = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$

a. M is an inverse matrix
b. M is not an inverse matrix
c. M is a nilpotent matrix
d. a, b and c are not true.

Prove: Let x is a vector of n elements, $x = (x_1, x_2, \ldots, x_n)^T$. Since $x^T M x = \sum_{k=1}^{n} (x_{k+1} - x_k)^2 \ge 0$ where $x_1 = x_{n+1}$. Thus M is a positive definite matrix, so M is inversible. M cannot be a nilpotent matrix.

10. Let E, F and G be three vector space in $\mathbb{K}$, $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$. Which of the following proposition is not true:

a. $Ker(gof) = f^{-1}(Ker(g))$
b. $Ker(gof) \supset Ker(f)$
c. $Im(gof) = g(Im(f))$
d. $Im(gof) \supset Im(g)$

Prove: For all x of E :

- $x \in Ker(gof) \Leftrightarrow (gof)(x) = 0 \Leftrightarrow f(x) \in Ker(g) \Leftrightarrow x \in f^{-1}(Ker(g))$

$$Ker(gof) = f^{-1}(Ker(g))$$

- Since $Ker(g) \supset \{0\}$, $Ker(gof) = f^{-1}(Ker(g)) \supset f^{-1}(\{0\}) = Ker(f)$
- $Im(gof) = (gof)(E) = g\big(f(E)\big) = g\big(Im(f)\big)$
- Since $Im(f) \subset F$, $Im(gof) = g\big(Im(f)\big) \subset g(F) = Im(g)$